Covariant cosmological perturbation dynamics in the large-scale limit

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Using the existence of a covariant conserved quantity on large perturbation scales in a spatially flat perfect fluid or scalar field universe, we present a general formula for gauge-invariantly defined comoving energy density perturbations which encodes the entire linear perturbation dynamics in a closed time integral. On this basis we discuss perturbation modes in different cosmological epochs.

I. INTRODUCTION

Because of its notable significance for the early stages of structure formation in the universe cosmological perturbation theory has received considerable attention during the past decades (see, e.g. [1–4]).

Depending on the choice of basic perturbation variables there are different possibilities to formulate the dynamics of primordial inhomogeneities. These choices may refer to different gauges or to different sets of gauge-invariant variables. In this paper we focus on the covariant approach which uses exactly defined tensorial quantities ([5–8]) instead of conventional, generally gauge-dependend perturbation variables. A covariant approach is conceptionally superior to noncovariant approaches since it avoids the explicit introduction of a fictitious background universe and thus circumvents the gauge-problem, which just originates from the nonuniqueness of the conventional splitting of the spacetime into a homogeneous and isotropic zeroth order and first-order perturbations about this background. The basic dynamic quantity used by Olson [5], Woszczyna and Kulak [6], Ellis and Bruni [7] and Jackson [8] is the covariantly defined spatial gradient of the energy density of a comoving (with the four-velocity of the cosmic fluid) observer. The corresponding dimensionless, fractional quantity obeys a second-order differential equation (see, e.g. [8]).

As was shown recently [9], the linear cosmological perturbation theory of an almost homogeneous and isotropic universe may be simplified by introducing covariant variables defined with respect to hypersurfaces of constant expansion, constant curvature or constant energy density. These new quantities which represent suitable linear combinations of Ellis-Bruni-Jackson type variables turned out to be particularly useful to characterize a conserved quantity on large perturbation scales which was applied subsequently to study the perturbation dynamics in an inflationary universe. [10]. Generally, the importance of conserved quantities is well recognized in the literature ([11–16]). In this paper we clarify how the existence of a covariant conserved quantity may be used to solve the perturbation equations in the corresponding limit. Our main purpose is to derive a general formula which reduces the entire large-scale linear perturbation dynamics for arbitrary equations of state to a closed time integral over homogeneous "background" quantities. We then demonstrate how the cosmological mode structure for all cases of interest (exponential inflation, power-law inflation, radiation and matter dominated periods etc.) follow from the general formula in an elementary way.

This paper is organized as follows. Section II briefly recalls the basic relations of a covariant perturbation theory with special emphasis on the characterization of a conserved quantity in the large-scale limit. Section III is devoted to the general solution of the large-scale perturbation dynamics while section IV presents the detailed mode structure for a variety of cosmologically relevant equations of state. A short summary is given in section V.

II. BASIC RELATIONS OF A COVARIANT PERTURBATION THEORY

We consider the cosmic medium characterized by an energy-momentum tensor with perfect fluid structure,

$$T_{mn} = \rho u_m u_n + p h_{mn} \ , \qquad (m, n ... = 0, 1, 2, 3) \ , \qquad (1)$$

where ρ is the energy density, p is the pressure, u_m is the four-velocity ($u^m u_m = -1$), and $h_{mn} = g_{mn} + u_m u_n$ the spatial projector on surfaces orthogonal to the 4-velocity. We recall that the structure (1) is also valid for a minimally coupled scalar field with the identifications

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \ , \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \ , \qquad (2)$$

and

$$u_i = -\frac{\phi_{,i}}{\sqrt{-g^{ab}\phi_{,a}\phi_{,b}}} \ , \tag{3}$$

where $\dot{\phi} \equiv \phi_{,a}u^a = \sqrt{-g^{ab}\phi_{,a}\phi_{,b}}$ and $V(\phi)$ is the scalar field potential.

Local energy momentum conservation $T^{ik}_{\ ;k} = 0$, implies

$$\dot{\rho} = -\Theta\left(\rho + p\right) \ , \qquad \left(\rho + p\right)\dot{u}^m = -p_{,k}h^{mk} \ . \tag{4}$$

Here, $\Theta \equiv u^i_{\ ;i}$ is the fluid expansion and $\dot{u}^m \equiv u^m_{;n}u^n$ the fluid acceleration.

Additionally, we make use of the Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\left(\sigma^2 - \omega^2\right) - \dot{u}^a_{;a} - \Lambda + \frac{\kappa}{2}\left(\rho + 3p\right) = 0 \ , \tag{5}$$

where Λ is the cosmological constant and κ is Einstein's gravitational constant. The magnitudes of shear and vorticity are defined by

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} \ , \qquad \omega^2 \equiv \frac{1}{2}\omega_{ab}\omega^{ab} \ , \tag{6}$$

with

$$\sigma_{ab} = h^c_a h^d_b u_{(c;d)} - \frac{1}{3}\Theta h_{ab} \ , \qquad \omega_{ab} = h^c_a h^d_b u_{[c;d]} \ . \tag{7}$$

The 3-curvature scalar of the projected metric,

$$\mathcal{R} = 2\left(-\frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 + \kappa\rho + \Lambda\right) \ , \tag{8}$$

reduces to the 3-curvature of the surfaces orhogonal to u^a for $\omega = 0$. The homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) universes correspond to the special case $\sigma = \omega = \dot{u}^a = 0$.

Suitable covariant variables to characterize spatial inhomogeneities are [7–10]

$$D_a \equiv \frac{ah^c_a\rho_{,c}}{\rho + p} \ , \qquad P_a \equiv \frac{ah^c_a p_{,c}}{\rho + p} \ , \qquad t_a \equiv ah^c_a\Theta_{,c} \ , \tag{9}$$

where a is a length scale generally defined by $\Theta \equiv 3\dot{a}/a$. The quantities D_a and P_a represent fractional, comoving (with the fluid four-velocity) energy density and pressure perturbations, respectively. Inhomogeneities in the expansion are described by the quantity t_a.

In a homogeneous and isotropic FLRW universe (superscript 0) the Friedmann equation

$$\kappa\rho^{(0)} = \frac{1}{3}\left(\Theta^{(0)}\right)^2 + \frac{1}{2}\mathcal{R}^{(0)} - \Lambda \ , \tag{10}$$

holds and the Raychaudhuri equation (5) reduces to

$$\dot{\Theta}^{(0)} + \frac{3}{2}\kappa\left(\rho^{(0)} + p^{(0)}\right) = \frac{1}{2}\mathcal{R}^{(0)} \ . \tag{11}$$

The three-curvature in this case is known to be $\mathcal{R}^{(0)} = 6k/a^2$, where $k = 0, \pm 1$ and a now coincides with the scale factor of the Robertson-Walker metric.

In order to describe the dynamics of the inhomogeneities in terms of D_a, P_a and t_a we differentiate the first Eq. (4), project orthogonal to u_a and multiply by a. The left-hand side of the resulting equation may be written as (cf. [8])

$$ah^c_m\dot{\rho}_{,c} = h^a_m\left(ah^c_a\rho_{,c}\right)^{\cdot} - a\dot{u}_m\dot{\rho} + \left(\omega^c_{\ m} + \sigma^c_{\ m}\right)ah^n_c\rho_{,n} \ , \tag{12}$$

where we have used the well-known decomposition

$$u_{i;n} = -\dot{u}_i u_n + \sigma_{in} + \omega_{in} + \frac{\Theta}{3}h_{in}$$

of the covariant derivative of the four-velocity.

Restricting ourselves to first-order deviations from homogeneity and isotropy we find [9]

$$\dot{D}_\mu + \frac{\dot{p}}{\rho + p}D_\mu + t_\mu = 0 \ , \qquad (\mu, \nu ... = 1, 2, 3) \ . \tag{13}$$

Analogously, one obtains from Eq. (5),

$$\dot{t}_\mu = -\frac{2}{3}\Theta t_\mu - \frac{\kappa}{2}\left(\rho + p\right)D_\mu - \left(\frac{1}{2}\mathcal{R} + \frac{\nabla^2}{a^2}\right)P_\mu \ . \tag{14}$$

Combining the last two equations and defining the sound velocity c_s in a standard way as $c_s^2 = \dot{p}/\dot{\rho}$, the first-order inhomogeneities are governed by the second-order equation

$$\ddot{D}_\mu + \left(\frac{2}{3} - c_s^2\right)\Theta\dot{D}_\mu - \Big[\left(c_s^2\right)^{\cdot}\Theta + \left(\frac{\kappa}{2}\left(\rho - 3p\right) + 2\Lambda\right)c_s^2 + \frac{\kappa}{2}\left(\rho + p\right)\Big]D_\mu = \frac{\nabla^2}{a^2}P_\mu \ . \tag{15}$$

For a fluid (superscript f) one has $P_a^{(f)} = c_s^2 D_a^{(f)}$ and Eq. (15) corresponds to Jackson's [8] equation (57). For a scalar field (superscript s), because of $h^c_a\phi_{,c} = 0$, the potential term neither contributes to $D_a^{(s)}$ nor to $P_a^{(s)}$ and, consequently, $P_a^{(s)} = D_a^{(s)}$ is valid [10] which, if used in Eq. (15), results in a closed equation for D_μ as well.

The well-known second-order differential equation (15) together with a corresponding expression for P_μ provides us with a complete description of linear scalar perturbations in perfect fluid and scalar field universes. Our aim here is to find the general solution of equation (15) in the large-scale limit for spatially flat ($k = 0$) cosmological models. To this purpose we rewrite the perturbation dynamics in terms of a different basic variable.

III. SOLVING THE LARGE-SCALE PERTURBATION DYNAMICS

As follows from the definitions (9), the quantities D_a, P_a and t_a are defined with respect to comoving hypersurfaces. It was shown in [9] that the perturbation dynamical description simplifies if written in terms of covariant

variables defined with respect to hypersurfaces of constant curvature, constant expansion, or constant energy density. These variables correspond to certain combinations of Ellis-Bruni-Jackson type variables. For example, the exactly defined covariant quantity

$$D_a^{(ce)} \equiv D_a - \frac{\dot{\rho}}{\rho + p}\frac{t_a}{\dot{\Theta}} \tag{16}$$

represents in first order the fractional, spatial gradient of the energy density on hypersurfaces of constant expansion (superscript ce) [9,10]. Rewriting the linear perturbation dynamics in terms of $D_a^{(ce)}$ and restricting ourselves to $k = 0$, one obtains

$$\left[a^2 \dot{\Theta} D_\mu^{(ce)}\right]^{\cdot} = -a^2 \Theta \frac{\nabla^2}{a^2} P_\mu \ . \tag{17}$$

Except for the spatial pressure gradient on the right-hand side of Eq. (17) all terms of the dynamical perturbation equation may be included into a first time derivative. With $D_\mu = D_{(m)} \nabla_\mu Q_{(m)}$, $P_\mu = P_{(m)} \nabla_\mu Q_{(m)}$ and $D_\mu^{(ce)} = D_{(m)}^{(ce)} \nabla_\mu Q_{(m)}$ (and corresponding relations for the other perturbation quantities), where the $Q_{(m)}$ satisfy the Helmholtz equation $\nabla^2 Q_{(m)} = -m^2 Q_{(m)}$, the quantity m is related to the physical wavelength by $\lambda = 2\pi a/m$ ([1,16–18]). It follows that the spatial gradient terms on the right-hand side of Eq.(17) may be neglected on large perturbation scales ($m \ll 1$) and the quantity $a^2 \dot{\Theta} D_{(m)}^{(ce)}$ is a conserved quantity both for a perfect fluid and a scalar field.

Denoting the conserved quantity by $-E_{(m)}$, i.e.

$$a^2 \dot{\Theta} D_{(m)}^{(ce)} \equiv -E_{(m)} = const \ , \qquad (m \ll 1) \ , \tag{18}$$

and taking into account the relation

$$\frac{\dot{\Theta}}{\Theta} D_\mu^{(ce)} = -\dot{D}_\mu + \left[\frac{\dot{\Theta}}{\Theta} + c_s^2 \Theta\right] D_\mu \tag{19}$$

between $D_\mu^{(ce)}$ and D_μ which follows from the definition (16) of $D_\mu^{(ce)}$ and Eq. (13), the equation to solve is

$$\dot{D}_{(m)} - \left(\frac{\dot{\Theta}}{\Theta} + c_s^2 \Theta\right) D_{(m)} = \frac{E_{(m)}}{a^2 \Theta} \ , \qquad (m \ll 1) \ . \tag{20}$$

With $\Theta = 3\dot{a}/a$ and the abbreviation

$$q \equiv \int \left(c_s^2\right)^{\cdot} \ln a^3 dt \tag{21}$$

the general solution to Eq. (20) for $m \ll 1$ is

$$D_{(m)} = \Theta a^{3c_s^2} \exp\{-q\} \cdot \cdot \left[\int^t dt \left(\frac{E_{(m)}}{\Theta^2 a^{2+3c_s^2}}\right) \exp\{q\} + C_{(m)}\right] \ , \tag{22}$$

where $C_{(m)}$ is an integration constant. For many purposes it is sufficient to consider $c_s^2 \approx const$ in which case $q = 0$ and the solution (22) reduces to

$$D_{(m)} = \Theta a^{3c_s^2} \left[\int^t dt \left(\frac{E_{(m)}}{\Theta^2 a^{2+3c_s^2}}\right) + C_{(m)}\right] \ , \quad (m \ll 1) \, . \tag{23}$$

Formula (22) comprises the entire large-scale linear perturbation dynamics for arbitrary equations of state. It is the main result of this paper. In the following section we use the expression (22) (or the special case (23)) to derive the mode structure for different cosmologically relevant equations of state.

IV. COSMOLOGICAL MODE STRUCTURE IN THE LARGE-SCALE LIMIT

A. Vanishing cosmological constant ($\Lambda = 0$)

Let us assume equations of state $p = (\gamma - 1)\rho$ with constant values of γ in the range $0 \leq \gamma \leq 2$, i.e. $c_s^2 \equiv \dot{p}/\dot{\rho} = \gamma - 1$. The case $\gamma = 0$, equivalent to $p = -\rho$ and $c_s^2 = -1$, is characterized by $\rho = const$, $\Theta \equiv 3H = const$, i.e. $a \propto \exp[Ht]$. Use of these dependences in formula (23) yields a behaviour $\propto a^{-2}$ and $\propto a^{-3}$ for the cosmological modes. Any perturbation about the de Sitter spacetime is exponentially damped [10].

For $0 < \gamma \leq 2$ we have $\rho \propto a^{-3\gamma}$ and, via Eq. (10), $a \propto t^{2/(3\gamma)}$. Since $\Theta \propto t^{-1}$ one obtains a behaviour $\propto a^{3\gamma-2} \propto t^{2-4/(3\gamma)}$ and $\propto a^{3\gamma/2-3} \propto t^{1-2/\gamma}$ for the large-scale modes [8] which coincides with corresponding results in [19].

It follows that in the range $0 < \gamma < 2/3$ where $a \propto t^n$ with $n > 1$ (power-law inflation) both perturbation modes are decaying. The case $\gamma = 2/3$, corresponding to so-called K-matter [20], is characterized by $a \propto t$. The dominant mode turns out to be constant in this case [21] while the second mode decays as $a^{-2} \propto t^{-2}$.

In the interval $2/3 < \gamma < 2$ there is one decaying and one growing mode. For $\gamma = 1$ (dust) one reproduces the well-known behaviour $\propto a \propto t^{2/3}$ and $\propto a^{-3/2} \propto t^{-1}$ for the large-scale perturbation modes, while one obtains modes $\propto a^2 \propto t$ and $\propto a^{-1} \propto t^{-1/2}$ for $\gamma = 4/3$ (radiation).

For $\gamma = 2$ (stiff matter) there is no decaying mode. The first term in (23) grows as $a^4 \propto t^{4/3}$ while the second one remains constant.

None of these results is new. Our point here is to demonstrate the possibility of a unified representation of the cosmological mode structure via formulae (22) or (23). We believe this representation to be useful also under more general circumstances, where, e.g. c_s^2 varies in time or in multifluid cosmological models to be studied elsewhere.

B. Λ-dominated universe

For $\Lambda \gg \kappa\rho$ in Eq. (10) we have $\Theta = 3H = const$, i.e. $a \propto \exp[Ht]$. Matter on this background is again characterized by $p = (\gamma - 1)\rho$ with $c_s^2 = (\gamma - 1)$. ¿From the general formula (23) we find perturbation modes $\propto \exp[-2Ht]$ and $\propto \exp[3(\gamma - 1)]$. While the first mode exhibits the same behaviour (exponential damping) as the dominating mode for $\Lambda = 0$ and $\gamma = 0$, the behaviour of the second mode strongly depends on γ. It decays exponentially for any $\gamma < 1$, e.g. for K-matter [21] with $\gamma = 2/3$, but grows for $\gamma > 1$, e.g. for radiation with $\gamma = 4/3$. Comoving perturbations in relativistic matter in a Λ-dominated universe are exponentially unstable. A behaviour such as this was first found in [22] by directly solving a second-order equation of the type (15). A Λ-dominated universe differs from the previous ones with $\Lambda = 0$ in so far as the background dynamics according to (10) is completely determined by Λ in the case considered here ($\kappa\rho \ll \Lambda$). On the other hand, differentiating Eq. (5) in order to obtain Eq. (14) the cosmological constant drops out and does not enter the perturbation dynamics explicitly. There are perturbations only in the component which is dynamically negligible in the background but not, by definition, in Λ which, however, determines the evolution of a and Θ in formula (23). The mentioned instability refers to this kind of perturbations.

V. SUMMARY

We have presented a compact, covariant version of linear cosmological perturbation theory for media characterized by an energy-momentum tensor of the perfect fluid type. The general solution of the large-scale perturbation dynamics in a spatially flat universe was obtained as a closed time integral for comoving energy density perturbations, resulting in a comprehensive picture of the cosmic mode structure for different cosmological epochs including exponential and power-law inflation as well as a K-matter period, the standard FLRW cases and matter perturbations in a Λ-dominated universe.

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